

# The Critical Parameter for the Heat Equation with a Noise Term to Blow Up in Finite Time

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## Abstract

Consider the stochastic partial differential equation

$$u_t = u_{xx} + u^\gamma \dot{W},$$

where  $x \in \mathbf{I} \equiv [0, J]$ ,  $\dot{W} = \dot{W}(t, x)$  is 2-parameter white noise, and we assume that the initial function  $u(0, x)$  is nonnegative and not identically 0. We impose Dirichlet boundary conditions on  $u$  in the interval  $\mathbf{I}$ . We say that  $u$  blows up in finite time, with positive probability, if there is a random time  $T < \infty$  such that

$$P\left(\limsup_{t \uparrow T} \sup_x u(t, x) = \infty\right) > 0.$$

It was known that if  $\gamma < 3/2$ , then with probability 1,  $u$  does not blow up in finite time. It was also known that there is a positive probability of finite time blow-up for  $\gamma$  sufficiently large.

In this paper, we show that if  $\gamma > 3/2$ , then there is a positive probability that  $u$  blows up in finite time.

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# 1 Introduction

We consider the heat equation with a nonlinear additive noise term.

$$\begin{aligned} u_t &= u_{xx} + u^\gamma \dot{W}, & t > 0, x \in \mathbf{I} \equiv [0, J] \\ u(t, 0) &= u(t, J) = 0 \\ u(0, x) &= u_0(x) \end{aligned} \tag{1.1}$$

Here,  $\dot{W} = \dot{W}(t, x)$  is 2-parameter white noise,  $\gamma \geq 1$ , and  $u_0(x)$  is a continuous nonnegative function on  $\mathbf{I}$ , vanishing at the endpoints, but not identically zero. Suppose that we are working on a probability space  $(\Omega, \mathcal{F}, P)$ , and fix a point  $\omega \in \Omega$ . If there exists a random time  $T = T(\omega) < \infty$  such that

$$\limsup_{t \uparrow T} \sup_{x \in \mathbf{I}} u(t, x) = \infty$$

then we say that  $u$  blows up in finite time (for the point  $\omega$ ).

For deterministic partial differential equations, there is a large literature about blow-up in finite time. See [Fuj66], [FM85], [FM86], [FK92], and [LN92] for example. Suppose that we are dealing with the equation

$$\begin{aligned} \frac{\partial w(t, x)}{\partial t} &= \Delta w(t, x) + g(w(t, x)) \\ w(0, x) &= w_0(x). \end{aligned}$$

One basic idea is the following. Suppose that  $g(w)$  increases faster than linearly. If a high peak forms in the solution  $w(t, x)$ , then the term  $g(w(t, x))$  will win out over the term  $\Delta w(t, x)$ , the growth of the peak will be governed by the ordinary differential equation

$$w'(t) = g(w(t)).$$

We can solve this equation explicitly, and its solutions often blow up in finite time.

On the other hand, for stochastic partial differential equations (SPDE) there are very few papers about finite-time blow-up. Apart from the heat equation, the author [Mue97a] has studied the wave equation

$$\begin{aligned} \frac{\partial^2 u_t(x)}{\partial t^2} &= \Delta u_t(x) + g(u_t(x)) \dot{W}(t, x), & t > 0, x \in \mathbf{R} \\ \frac{\partial u_0(x)}{\partial t} &= h_1(x) \\ u_0(x) &= h_0(x). \end{aligned}$$

where  $g(u)$  grows like  $u(\log u)^\alpha$  for some  $\alpha > 1$ . For  $g(u) = u^\alpha$  with  $\alpha > 1$ , one would guess that solutions would blow up in finite time. But finite time blow-up is not known for any value of  $\alpha$ . Similar techniques were used in [Mue93] to obtain a modulus of continuity for solutions of the wave equation with noise in higher dimensions, with correlated Gaussian noise instead of white noise.

There is more precise information about the heat equation with noise. Suppose that  $u$  is a solution to (1.1). In [Mue91] it was shown that if  $\gamma < 3/2$ , then, with probability 1,  $u$  does not blow up in finite time. Krylov [Kry94] gave another proof of this fact for a more general class of equations. The papers [Mue97a] and [Mue97b] are also relevant. We refer the reader to Pardoux [Par93] for this and other questions about parabolic SPDE. Returning to the question of blow-up, it was shown in [MS93] that there exists  $\gamma_0 > 1$  such that if  $\gamma > \gamma_0$ , then with positive probability,  $u$  blows up in finite time. The argument in [MS93] was not sharp enough to give the best value of  $\gamma_0$ , and the question of whether one could take  $\gamma_0 = 3/2$  was left open. The main theorem of this paper answers this question in the affirmative.

**Theorem 1** *Let  $u(t, x)$  satisfy (1.1), and suppose that  $\gamma > 3/2$ . Then, with positive probability,  $u$  blows up in finite time.*

Of course, Theorem 1 does not tell us what happens at  $\gamma = 3/2$ . Surprisingly, the proof of Theorem 1 uses many of the same ideas as in [MS93], although in a sharper form.

Now we discuss the rigorous meaning of (1.1), following the formalism of Walsh [Wal86], chapter 3. Before giving details, we set up some notation. Let  $G(t, x, y)$  be the fundamental solution of the heat equation on  $\mathbf{I}$ . If  $G(t, x)$  is written as a function of 2 variables, we let  $G(t, x)$  be the fundamental solution of the heat equation on  $\mathbf{R}$ . In other words

$$G(t, x) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right).$$

It is well known that

$$G(t, x, y) \leq G(t, x - y).$$

We regard (1.1) as shorthand for the following integral equation.

$$u(t, x) = \int_{\mathbf{I}} G(t, x, y) u_0(y) dy + \int_0^t \int_{\mathbf{I}} G(t - s, x, y) g(u(s, y)) W(dy ds) \quad (1.2)$$

where the final term in (1.2) is a white noise integral in the sense of [Wal86], Chapter 2. Because  $g(u)$  is locally Lipschitz, standard arguments show that (1.1) has a unique solution  $u(t, x)$  valid up to the time  $\sigma_L$  at which  $|u(t, x)|$  first reaches the level  $L$  for some  $x \in \mathbf{I}$ . Similar arguments are given in [Wal86], Theorem 3.2 and Corollary 3.4, and his reasoning easily carries over to our case. Letting  $L \rightarrow \infty$ , we find that (1.1) has a unique solution for  $t < \sigma$ , where  $\sigma = \lim_{L \rightarrow \infty} \sigma_L$ . If  $\sigma < \infty$ , one has

$$\limsup_{t \uparrow \sigma} \sup_{x \in \mathbf{I}} |u(t, x)| = \infty.$$

Our goal is to show that  $\sigma = \infty$  with probability 1.

More generally, we regard

$$\begin{aligned} v_t &= v_{xx} + g(v)\dot{W}, & t > 0, x \in \mathbf{I} \\ v(t, 0) &= v(t, J) = 0 \\ v(0, x) &= v_0(x) \end{aligned}$$

as a shorthand for the following integral equation, which may only be valid up to some blow-up time.

$$v(t, x) = \int_{\mathbf{I}} G(t, x, y) v_0(y) dy + \int_0^t \int_{\mathbf{I}} G(t - s, x, y) g(v(s, y)) W(dy ds)$$

Lastly, we will always work with the  $\sigma$ -fields  $\mathcal{F}_t = \mathcal{F}_t^W$  generated by the white noise up to time  $t$ . That is,  $\mathcal{F}_t$  is the  $\sigma$ -field generated by the random variables  $\int_0^t \int_{\mathbf{I}} \phi(s, x) W(dx ds)$ , where  $\phi$  varies over all continuous functions on  $[0, t] \times \mathbf{I}$ .

We now summarize the argument in [MS93], which is based on the analysis of the formation of high peaks. Such peaks will occur with positive probability. We wish to show that, with positive probability, such peaks grow until they blow up in finite time. If a high peak forms, we rescale the equation and divide the mass of the peak into a collection of peaks of smaller mass, and these peaks evolve almost independently. In this way we compare the evolution of  $u$  to a branching process. Large peaks are regarded as particles in this branching process. Offspring are peaks which are higher by some factor. We show that the expected number of offspring is greater than one when  $\gamma > 3/2$ , and thus the branching process survives with positive probability, corresponding to blowup in finite time.

Finally, we remark that in (1.1), we could replace  $u^\gamma$  with a function  $g(u)$  satisfying  $g(u) > cu^\gamma$  for some  $c > 0$ . Then Theorem 1 would still hold, provided  $\gamma > 3/2$ .

## 2 Proof of Theorem 1

We give a proof by contradiction. Assume that

$$P(\sigma < \infty) = 0. \quad (2.1)$$

We recall Lemma 2.4 of [MS93].

**Lemma 1** *Suppose that  $u$  solves (1.1) up to some  $\mathcal{F}_t^W$  stopping time  $\tau$ . Let  $\bar{L} > 0$ . If we let*

$$\tilde{v}(t, x) \equiv \bar{L}^{-1} u\left(t\bar{L}^{4(1-\gamma)}, x\bar{L}^{2(1-\gamma)}\right) \quad t \geq 0, \quad x \in \mathbf{I}\bar{L}^{2(\gamma-1)}$$

*then  $\tilde{v}(t, x)$  solves*

$$\begin{aligned} \frac{\partial \tilde{v}}{\partial t} &= \frac{\partial^2 \tilde{v}}{\partial x^2} + b(\tilde{v}, \tilde{\xi}) \dot{\tilde{W}} \\ \tilde{v}(t, 0) &= \tilde{v}\left(t, J\bar{L}^{2(\gamma-1)}\right) = 0 \\ \tilde{v}(0, \cdot) &= \tilde{v}_0 \quad t \geq 0, \quad 0 \leq x \leq J\bar{L}^{2(\gamma-1)} \end{aligned}$$

*up to the  $\mathcal{F}_t^{\tilde{W}}$ -stopping time  $\tau\bar{L}^{2(1-\gamma)}$ , where  $\tilde{v}_0(x) \equiv \bar{L}^{-1}u_0\left(x\bar{L}^{2(1-\gamma)}\right)$  for all  $0 \leq x \leq J\bar{L}^{2(\gamma-1)}$ , for all  $t \geq 0$  and  $0 \leq x \leq J\bar{L}^{2(\gamma-1)}$ , and  $\tilde{W}$  is the white noise on  $\mathcal{B}(\mathbf{R}_+ \times [0, J\bar{L}^{2(\gamma-1)}])$  defined by*

$$\tilde{W}(A) \equiv \bar{L}^{3(\gamma-1)} \int_{\mathbf{R}_+ \times \mathbf{I}} \chi_A(t\bar{L}^{4(\gamma-1)}, x\bar{L}^{2(\gamma-1)}) W(dt, dx)$$

*for all  $A$  in  $\mathcal{B}(\mathbf{R}_+ \times [0, J\bar{L}^{2(\gamma-1)}])$  with finite Lebesgue measure.*

In [MS93] we fixed  $\bar{L}$ , which we called  $L$ , and took  $\gamma$  to be very large. In the current proof we wish to deal with all  $\gamma > 3/2$ , so we take  $\bar{L}$  as our large parameter. We will find that the probability of a peak getting up to level  $\bar{L}$  is about  $p = 1/\bar{L}$ , from the gambler's ruin problem. Using Lemma 1, we will see that after rescaling, a peak of size  $L$  gives rise to  $N = L^{2(\gamma-1)}$  offspring. Thus, the expected number of offspring of our initial peak should be

$$pN = (1/L) \cdot L^{2(\gamma-1)} = L^{2\gamma-3}.$$

If  $\gamma > 3/2$ , then  $2\gamma - 3 > 0$  and  $pN \rightarrow \infty$  as  $L \rightarrow \infty$ . Of course, the above heuristic calculation will suffer from the rough estimates we make during

the course of the proof. Our hope is that taking  $pN$  is large enough will compensate for all of our sloppiness.

We will consider solutions  $\bar{u}(t, x)$  to a slightly more general equation than (1.1).

$$\begin{aligned}\bar{u}_t &= \bar{u}_{xx} + g(\bar{u})\dot{W}, & t > 0, x \in \mathbf{I} \equiv [0, J] \\ \bar{u}(t, 0) &= \bar{u}(t, J) = 0 \\ \bar{u}(0, x) &= u_0(x)\end{aligned}\tag{2.2}$$

where  $g : [0, \infty) \rightarrow [0, \infty)$  is a locally Lipschitz function satisfying  $g(0) = 0$  and  $g(u) \geq u^\gamma$  for  $u > 0$ . The same argument as for (1.1) gives existence and uniqueness of  $\bar{u}(t, x)$  up to the blow-up time for  $\bar{u}$ .

Let  $\varphi(t, x) = \varphi^{(T)}(t, x)$  be a solution of the backward heat equation

$$\varphi_t = -\varphi_{xx} \quad 0 \leq t \leq T, x \in \mathbf{R} \tag{2.3}$$

with “final condition”

$$\varphi(T, x) = \frac{1}{\sqrt{4\pi T}} \exp\left(-\frac{x^2}{4T}\right).$$

Of course,  $\varphi(T, x)$  is the heat kernel evaluated at time  $T$ , and therefore, for  $0 \leq t \leq T$ ,

$$\varphi(t, x) = \frac{1}{\sqrt{4\pi(2T-t)}} \exp\left(-\frac{x^2}{4(2T-t)}\right).$$

Next, a short calculation yields the following lemma.

**Lemma 2** *Let  $\varphi(t, x)$  be as in (2.3). If  $0 \leq t \leq T$ , then*

$$\varphi^{(T)}(t, x) \geq \sqrt{2}\varphi^{(T)}(T, x)$$

**Proof of Lemma 2.** Since  $0 \leq t \leq T$ , we have that  $T/(2T-t) \geq 1/2$ , and  $T^{-1} \geq (2T-t)^{-1}$ . Therefore

$$\begin{aligned}\frac{\varphi^{(T)}(t, x)}{\varphi^{(T)}(T, x)} &= \sqrt{\frac{T}{2T-t}} \exp\left(\frac{x^2}{4} [T^{-1} - (2T-t)^{-1}]\right) \\ &\geq \sqrt{2}\end{aligned}$$

This proves Lemma 2. ■

For future use, we also compute the  $\mathbf{L}^1$  norm of  $\varphi(t, x)^a$ , for  $a > 0$ . We claim that there exists a constant  $C = C(a) > 0$ , not depending on  $T$ , such that

$$\begin{aligned}
\|\varphi(t, x)^a\|_1 &= \int_{\mathbf{I}} \varphi(t, x)^a dx \\
&= (4\pi(2T - t))^{-a/2} \int_{\mathbf{I}} \exp\left(-\frac{ax^2}{4(2T - t)}\right) dx \\
&= C'(a)(2T - t)^{(1-a)/2} \\
&\leq C(a)T^{(1-a)/2}.
\end{aligned} \tag{2.4}$$

Now, let

$$M(t) = \int_{\mathbf{I}} \varphi(t, x) \bar{u}(t, x) dx, \quad 0 \leq t \leq T.$$

and note that  $M(t)$  is a continuous  $\mathcal{F}_t$  martingale for  $0 \leq t \leq T$ . This assertion was proven in Lemma 2.3 of [MS93]. One can also check it heuristically, by formally differentiating  $M(t)$  and applying (1.1) and (2.3). Lemma 2.3 of [MS93] also states that  $0 \leq t \leq T$ ,  $M(t)$  has square variation

$$\langle M \rangle_t = \int_0^t \int_{\mathbf{I}} g(\bar{u}(s, x)) \varphi(s, x)^2 dx ds \geq \int_0^t \int_{\mathbf{I}} \bar{u}(s, x)^{2\gamma} \varphi(s, x)^2 dx ds. \tag{2.5}$$

Of course,  $M(t) = M^T(t)$  implicitly depends on  $T$ . We now prove the following lower bound on  $\langle M \rangle_t$ .

**Lemma 3** *There exists a constant  $C_1 > 0$ , not depending on  $T$ , such that if  $0 \leq t \leq T$ , then*

$$\langle M \rangle_t \geq C_1 T^{-1/2} \int_0^t M(s)^{2\gamma} ds.$$

**Proof of Lemma 3.**

Let

$$a = \frac{2\gamma - 2}{2\gamma - 1}. \tag{2.6}$$

Note that

$$\frac{2 - a}{2\gamma} + a = 1 \tag{2.7}$$

and

$$\frac{1 - a}{2} \cdot (1 - 2\gamma) = -\frac{1}{2}. \tag{2.8}$$

Furthermore, for  $t$  fixed,

$$\frac{\varphi(t, x)^a}{\|\varphi(t, x)^a\|_1}$$

is a probability density over  $x \in \mathbf{R}$ . Using Jensen's inequality, (2.7), (2.4), and (2.8), we find that

$$\begin{aligned} & \int_{\mathbf{I}} \bar{u}(s, x)^{2\gamma} \varphi(s, x)^2 dx \\ &= \|\varphi(s, x)^a\|_1 \int_{\mathbf{I}} \bar{u}(s, x)^{2\gamma} \varphi(s, x)^{2-a} \frac{\varphi(s, x)^a}{\|\varphi(s, x)^a\|_1} dx \\ &\geq \|\varphi(s, x)^a\|_1 \left( \int_{\mathbf{I}} \bar{u}(s, x) \varphi(s, x)^{(2-a)/(2\gamma)} \frac{\varphi(s, x)^a}{\|\varphi(s, x)^a\|_1} dx \right)^{2\gamma} \\ &= \|\varphi(s, x)^a\|_1^{1-2\gamma} \left( \int_{\mathbf{I}} \bar{u}(s, x) \varphi(s, x) dx \right) \\ &\geq \left( C(a) T^{(1-a)/2} \right)^{1-2\gamma} M(s)^{2\gamma} \\ &= C_1 T^{-1/2} M(s)^{2\gamma}. \end{aligned} \tag{2.9}$$

where  $C_1 = C(a)^{1-2\gamma}$ , and  $a$  was defined in (2.6). After integrating (2.9) over  $s \in [0, t]$  and putting this together with (2.5), we get Lemma 3. ■

Using Lemma 3, it is possible to compare  $M(t)$  to a time-changed Brownian motion. In the standard way, the new time scale is given by  $\langle M \rangle_t$ . Let

$$T(L) = 16C_1^{-2}L^8$$

and consider the following gambler's ruin problem. Start with  $M(0) = 2$ . Let  $\tau = \tau(L)$  be the first time  $t$  that  $M(t) = 1$  or  $M(t) = L$ . Using the optional sampling theorem in the usual way, we deduce that  $EM(\tau) = 2$ , and therefore (if  $M(0) = 2$ ),

$$P(M(\tau) = L) = \frac{1}{L-1}. \tag{2.10}$$

In fact, we wish to show

**Lemma 4** *If  $T = T(L) = 16C_1^{-2}L^8$ , then*

$$P(M(\tau \wedge T) = L) \geq \frac{1}{2(L-1)}.$$



**Proof of Lemma 4.**

The definition of  $\tau$  implies that for all  $t \in [0, \tau]$ , we have  $M(t) \geq 1$ . Therefore, by Lemma 3, if  $t \in [0, \tau]$  then

$$\langle M \rangle_t \geq C_1 T^{-1/2} t.$$

Now  $M(t)$  is a continuous supermartingale, so it follows that  $M(t)$  is greater than or equal to a time-changed Brownian motion with time scale  $\langle M \rangle_t$ . In other words, for some Brownian motion  $B(t)$ , we have  $M(t) \geq 2 + B(\langle M \rangle_t)$ . Therefore, since

$$\langle M \rangle_T \geq C T^{-1/2} T = C T^{1/2},$$

we have

$$\begin{aligned} P(T < \tau) &= P(T < \tau \leq \sigma, 1 < M(t) < L \text{ for } t \in [0, T]) \\ &\leq P(T < \tau, 1 < 2 + B(\langle M \rangle_t) < L \text{ for } 0 \leq t \leq T) \\ &= P(T < \tau, 1 < 2 + B(t) < L \text{ for } 0 \leq t \leq \langle M \rangle_T) \\ &= P(T < \tau, 1 < 2 + B(t) < L \text{ for } 0 \leq t \leq C_1 T^{1/2}) \\ &\leq P\left(\sup_{t \in [0, C_1 T^{1/2}]} B(t) < L - 2\right). \end{aligned}$$

Using the reflection principle, we continue with

$$\begin{aligned} P(T < \tau) &\leq 1 - P\left(\sup_{t \in [0, C_1 T^{1/2}]} B(t) \geq L - 2\right) \\ &= 1 - 2P\left(B(C_1 T^{1/2}) \geq L - 2\right) \\ &= P\left(|B(C_1 T^{1/2})| \leq L - 2\right) \\ &= \int_{-(L-2)}^{L-2} (2\pi C_1 T^{1/2})^{-1/2} \exp\left(-\frac{x^2}{2C_1 T^{1/2}}\right) dx \\ &\leq 2(L-2)(2\pi C_1 T^{1/2})^{-1/2} \\ &\leq C_1^{-1/2} L T^{-1/4}. \end{aligned}$$

Therefore, if

$$T = T(L) = 16C_1^{-2} L^8.$$

then

$$P(T < \tau) < \frac{1}{2(L-1)}.$$

Then, by (2.10),

$$\begin{aligned} P(M(\tau \wedge T) = L) &\geq P(M(\tau \wedge T) = L, T \geq \tau) \\ &= P(M(\tau) = L, T \geq \tau) \\ &= P(M(\tau) = L) - P(M(\tau) = L, T < \tau) \\ &\geq P(M(\tau) = L) - P(T < \tau) \\ &\geq \frac{1}{L-1} - \frac{1}{2(L-1)} \\ &= \frac{1}{2(L-1)}. \end{aligned}$$

This proves Lemma 4.  $\blacksquare$

Now we continue with the proof of Theorem 1. Lemma 4 implies that

$$p \geq \frac{1}{2(L-1)} \geq (2L)^{-1}$$

Using Lemma 1, with  $\bar{L} = L/2$ , we deduce that

$$N \geq K^{-1}(L/2)^{2(\gamma-1)} - 1 \geq K^{-1}(L/4)^{2(1-\gamma)}$$

if  $L$  is large enough. Thus we find that

$$pN \geq K^{-1}(4L)^{-1}(L/4)^{2(\gamma-1)} = K^{-1}4^{1-2\gamma}L^{2(\gamma-3/2)} > 1 \quad (2.11)$$

if  $\gamma > 3/2$  and  $L$  is large enough.

To finish the proof, we can apply the same argument as in [MS93], sections 3 and 4. Since these arguments carry over, word for word, we will merely summarize the argument here, and refer the reader to [MS93] for details.

First, we need to split up the solution  $u$ . For this, we quote Lemma 2.5 of [MS93]. But first, define

$$b(x, y) \equiv \sqrt{(x+y)^{2\gamma} - y^{2\gamma}}.$$

**Lemma 5** For  $t \geq 0$ ,  $x \in \mathbf{I}$ ,  $i = 1, 2, \dots, N$  consider the  $N$  recursively defined equations

$$\begin{aligned} \frac{\partial u^i}{\partial t} &= \frac{\partial^2 u^i}{\partial x^2} + b\left(u^i, \sum_{j=1}^{i-1} u^j\right) \dot{W}^i \\ u^i(t, 0) &= u^i(t, J) = 0 \\ u^i(0, \cdot) &= u_0^i. \end{aligned} \tag{2.12}$$

where  $u^0 \equiv 0$  by definition. Here the  $\{W^i\}$ 's are independent white noises and the  $\{u_0^i\}$  are some collection of nonnegative initial functions. Let us then define the process

$$\tilde{u}(t, \cdot) \equiv \begin{cases} \sum_{i=1}^N u^i(t, \cdot) & \text{for } 0 \leq t < \min\{\sigma(u^i) : i = 1, 2, \dots, N\} \\ \infty & \text{otherwise} \end{cases}$$

for all  $t \geq 0$ . Here,  $\sigma(u^i)$  denotes the blow-up time  $\sigma$  with respect to  $u^i$ . For  $t \geq 0$ ,  $x \in \mathbf{I}$ , we have that  $\tilde{u}$  is a solution of

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial t} &= \frac{\partial^2 \tilde{u}}{\partial x^2} + \tilde{u}^\gamma \dot{\tilde{W}} \\ \tilde{u}(t, 0) &= \tilde{u}(t, J) = 0 \\ \tilde{u}(0, \cdot) &= \sum_{i=1}^N u_0^i \end{aligned}$$

for some white noise  $\tilde{W}$  which is a linear combination of the  $\{W^i\}$ .

We use Lemma 5 to split up the solution  $u$  into the sum of solutions  $u^i$ . Later, we will further split up the  $u^i$ . Section 4 of [MS93] explains how to use Lemma 5 to split up  $u$  over and over again, at a sequence of stopping times. Each of these smaller solutions will have a larger noise term than in (1.1), so the corresponding total mass martingales  $U^i(t) = \int_{\mathbf{I}} u^i(t, x) dx$  will have

$$\langle U^i \rangle_t \geq \int_0^t U^i(s)^{2\gamma} ds.$$

We need a way to split up  $u$ , given that a certain integral is sufficiently large. The following lemma is an easy modification of Proposition 3.2 of [MS93].

**Lemma 6** Let  $E_+^\infty(\bar{J})$  denote the class of nonnegative  $\mathbf{C}^\infty$  functions on  $[0, \bar{J}]$ . There exists a constant  $K > 0$  such that the following holds. Let

$J > 4$  be fixed. Set  $\bar{J} \equiv J2^{2(\gamma-1)}$ . If  $N > 0$  is an integer, and  $f_0 \in E_+^\infty(\bar{J})$  satisfies

$$\int_0^{\bar{J}} \phi(t, x2^{2(1-\gamma)}; z_0, J) f_0(x) dx > KN,$$

for some  $z_0$  in  $[1, J-1]$  and some  $0 \leq t \leq 1$ , then there are functions  $\{f_i : i = 1, 2, \dots, N\} \subset E_+^\infty(\bar{J})$  such that

$$f_0 = \sum_{i=1}^N f_i$$

and for each  $i = 1, 2, \dots, N$ ,

$$\int_0^{\bar{J}} \phi(0, x; z_i, \bar{J}) f_i(x) dx \geq 2. \quad (2.13)$$

for some  $z_i$  in  $[1, \bar{J}-1]$

In [MS93], Lemma 6 was shown for  $N = \lfloor 2^{2\gamma-3} \rfloor$ , but the proof given there also implies the above result.

Now we continue the main argument. We can assume without loss of generality that

$$\int_{\mathbf{I}} G(T, x, y) u(0, y) dy \geq 2.$$

If this condition fails, wait until time 1, when it has a positive probability of holding. Now wait until time  $T$ . By Lemma 2.10, we have that

$$P\left(\int_{\mathbf{I}} G(2T, x, y) u(0, y) dy \geq L\right) \geq \frac{1}{2(L-1)} = p \quad (2.14)$$

Let

$$N = K^{-1} L^{2(\gamma-1)}. \quad (2.15)$$

Now perform the scaling as in Lemma 1, with  $\bar{L} = L/2$ . For the scaled function  $\tilde{v}$ , we see that

$$\int_{\mathbf{I}} G(2T, x, y) u(0, y) dy \geq L^{2(\gamma-1)} = KN.$$

Then, Lemma 6 shows that we can decompose

$$u(t, x) = \sum_{i=1}^N f_i(x)$$

such that for some set of points  $\{z_i\}_{i=1}^N$ ,

$$\int_{\mathbf{I}} G(2T, x, y) f_i(y) dy \geq 2.$$

We use these  $f_i$  as initial conditions for new functions  $u^i(t, x)$ , which satisfy (2.12), and we call these  $u^i(t, x)$  offspring of  $u(t, x)$ .

If

$$\int_{\mathbf{I}} G(2T, x, y) u(0, y) dy < L,$$

then we say that mass has died.

Repeating the argument, we find that there is mass alive at stage  $k$  if the branching process of the  $u$ 's is alive at stage  $k$ . But this is a Galton-Watson process with expected number of offspring at least

$$pN = K^{-1} L^{2(\gamma-1)} \frac{1}{2(L-1)} \geq 2^{-1} K^{-1} L^{2(\gamma-3/2)}$$

by (2.14) and (2.15). Therefore, if  $\gamma > 3/2$  and  $L$  is large enough, the expected number of offspring is at least

$$pN > 1$$

and there is a positive probability of survival. But survival means that there is mass present at each stage. This, in turn, means that  $u(t, x)$  blows up in finite time. Therefore, there is a positive probability of finite time blow-up.

But this conclusion contradicts our assumption (2.1) that  $P(\sigma < \infty) = 0$ . Thus, Theorem 1 is proved. ■

## References

- [FK92] S. Filippas and R. Kohn. Refined asymptotics for the blowup of  $u_t - \delta u = u^p$ . *Comm. Pure Appl. Math.*, 45:821–869, 1992.
- [FM85] A. Friedman and B. McLeod. Blow-up of positive solutions of semilinear heat equations. *Indiana U. Math. J.*, 34:425–447, 1985.
- [FM86] A. Friedman and B. McLeod. Blow-up of solutions of nonlinear degenerate parabolic equations. *Archive Rat. Mech. and Anal.*, 96:55–80, 1986.

- [Fuj66] H. Fujita. On the blowing up of solutions of the Cauchy problem for  $u_t = \delta u + u^{1+\alpha}$ . *J. Fac. Sci. Tokyo Sect. 1A Math.*, 13:109–124, 1966.
- [Kry94] N.V. Krylov. On  $L_p$ -theory of stochastic partial differential equations in the whole space. *SIAM J. Math. Anal.*, 27(2):313–340, 1994.
- [LN92] T.-Y. Lee and W.-M. Ni. Global existence, large time behavior and life span of solutions of a semilinear parabolic Cauchy problem. *Trans. Amer. Math. Soc.*, 33(1):365–378, 1992.
- [MS93] C. Mueller and R. Sowers. Blow-up for the heat equation with a noise term. *Prob. Th. Rel. Fields*, 97:287–320, 1993.
- [Mue91] C. Mueller. Long time existence for the heat equation with a noise term. *Prob. Th. Rel. Fields*, 90:505–518, 1991.
- [Mue93] C. Mueller. A modulus for the 3-dimensional wave equation with noise: dealing with a singular kernel. *Can. J. Math.*, 45(6):1263–1275, 1993.
- [Mue97a] C. Mueller. Long-time existence for signed solutions to the heat equation with a noise term. *Ann. Prob.*, 24(1):377–398, 1997.
- [Mue97b] C. Mueller. Long time existence for the wave equation with a noise term. *Ann. Prob.*, 25(1):133–152, 1997.
- [Par93] E. Pardoux. Stochastic partial differential equations, a review. *Bull. Sc. Math.*, 117:29–47, 1993.
- [Wal86] J.B. Walsh. An introduction to stochastic partial differential equations. In P. L. Hennequin, editor, *Ecole d’Ete de Probabilites de Saint Flour XIV-1984, Lecture Notes in Math. 1180*, Berlin, Heidelberg, New York, 1986. Springer-Verlag.